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Effective boundary conditions for Stokes flow over a rough surface

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Ensemble averaging combined with multiple scattering ideas is applied to the Stokes flow over a stochastic rough surface. The surface roughness is modelled by compact protrusions on an underlying smooth surface. It is established that the effect of the roughness on the flow far from the boundary may be represented by replacing the no-slip condition on the exact boundary by a partial slip condition on the smooth surface. An approximate analysis is presented for a sparse distribution of arbitrarily shaped protrusions and explicit numerical results are given for hemispheres. Analogous conclusions for the two-dimensional case are obtained. It is shown that in certain cases a traction force develops on the surface at an angle with the direction of the flow.

1. Introduction

Viscous flow over a complex boundary is a practical but difficult problem that can in general only be dealt with numerically. When the scale of interest is large compared with the scale of the surface features, however, it may be possible to account for the latter by means of an effective boundary condition imposed on a smooth surface approximating the actual one. In this case we may refer to the boundary as rough. One may distinguish two limit cases of rough surfaces: those of the 'wavy' and those of the 'bumpy' type. In the first case both the scale of surface irregularities and their slope are small. For bumpy surfaces, on the other hand, the surface slope remains finite irrespective of the smallness of the irregularities. In this paper we consider no-slip surfaces of the second type in which the roughness consists of bosses randomly distributed over a smooth surface. We assume that the bosses are sufficiently small to remain immersed in a region of fluid where the Stokes equations apply. Thus we explicitly rule out the much more difficult problem of protuberances significantly extending into the viscous sublayer or the turbulent boundary layer.

The general problem studied here has been considered before, especially for wavy surfaces. Nye's (1969, 1970) investigation was motivated by the mechanics of glacier sliding, while Richardson (1973) was concerned with the origin of the no-slip condition normally applied at solid boundaries. Surface roughness as a mechanism for inducing slip was investigated by Hocking (1976) and others (see Dussan V. 1979; Haley & Miksis 1991) in connection with the stress singularity associated with the motion of a contact line along a solid wall. The most recent studies on the subject are due to Miksis & Davis (1994), who allowed for the presence of a film of a different fluid

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coating the surface, and to Tuck & Kouzoubov (1995) who considered finite-slope and finite-Reynolds-number effects.

All of the previous studies are for wavy surfaces. As far as we know, bumpy surfaces of the type studied here have only been considered by Jansons in connection with the moving contact line problem (1985, 1986), the flow of a rarefied gas (1994), and slow viscous flow (1988). The last study addresses the same problem as considered here with the difference that free-slip 'microscopic' boundary conditions are imposed on the rough surface. Jansons's reliance on the method of images would prevent his technique from working in the no-slip case studied here. Furthermore, contrary to ours, his approach requires a locally uniform roughness distribution. As will be shown below, non-uniform distributions can give rise to some very interesting effects.

The approach that we use in this paper is suggested by the theory of multiple scattering (see e.g. Foldy 1945; Biot 1968; Ogilvy 1987, 1991; Twersky 1957, 1983) and we have recently applied it to the related problem of the Laplace equation in the presence of rough boundaries (Sarkar & Prosperetti 1995). It is based on the use of a local Green's function and ensemble averaging that can be used essentially in the same form for both two- and three-dimensional bosses.

In general, we find that when the length scale of interest is much larger than the characteristic size of the bosses, an effective boundary condition can be formulated which is however non-local (§4). This general expression can be given a local form with the additional assumption that the bosses be 'sparse', i.e. widely separated on average (§5). One can then give exact results for the case of identical hemispherical bosses (§6). Section 7 deals with the two-dimensional problem. An application of the results to Stokes flow past a rough sphere and to Poiseuille flow in a rough tube (§8) serves to elucidate their physical significance. In §8 we also consider an example of a new phenomenon arising in the presence of a spatially non-uniform boss distribution. We show that a traction can develop on an oscillating rough plate in the direction normal to that of the oscillations. This result also implies that such a plate would not fall vertically in a fluid, but would acquire a horizontal velocity component in its plane.

In conclusion we may cite a few other contexts in which partial slip boundary conditions arise, namely the flow next to the surface of a porous material (Beavers & Joseph 1967; Taylor 1971; Richardson 1971; Saffman 1971; Nield 1983), the effect of a wall on the flow of a suspension (Brunn 1981), and molecular diffusion along solid surfaces (Davis, Kezirian & Brenner 1994).

2. Formulation

We consider the viscous flow adjacent to a rough surface S_r consisting of N bosses with a characteristic linear dimension a randomly arranged over a smooth surface S_s. The surface S_r is rough in the sense that the smallest radius of curvature of S_s, R_s, is much larger than a (figure 1). For simplicity we take the bosses to have identical shape and orientation, a restriction that can be easily lifted as noted below in §3. A similar construction holds in the two-dimensional case in which S_s is a smooth line and the bosses are infinitely long ridges (see §7). On the rough surface S_r the velocity $\boldsymbol{u}$ satisfies the no-slip condition

$$\boldsymbol{u} = 0, \tag{2.1}$$

and our objective is to replace this no-slip condition valid on S_r by an approximate boundary condition on the underlying smooth surface S_s.

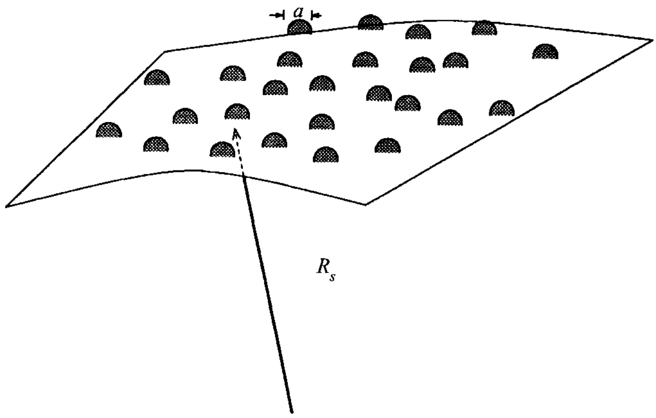

FIGURE 1. Model of an embossed surface

We assume that the mean shear near the surface S_r is small enough that the Stokes flow equations

$$\nabla p = \mu \nabla^2 \boldsymbol{u}, \qquad \nabla \cdot \boldsymbol{u} = 0, \tag{2.2}$$

are valid at least up to distances $\ell \gg a$ from the surface S_s. In the language of singular perturbations, the solution of (2.2) is the inner solution to be made unique by the condition of matching with an outer solution at a distance of the order of ℓ from the boundary. In the analysis that follows, we shall need a separation of scales such that $a \ll \ell \ll R_s$.

We start by decomposing the solution $(p, \boldsymbol{u})$ as

$$\boldsymbol{u} = \boldsymbol{u}_0 + \sum_{\alpha=1}^{N} \boldsymbol{v}^\alpha, \qquad p = p_0 + \sum_{\alpha=1}^{N} q^\alpha. \tag{2.3}$$

This decomposition is made unique by specifying that $(p_0, \boldsymbol{u}_0)$ satisfy the Stokes equations subject to the no-slip condition on the smooth surface S_s and to the matching condition. The fields $(q^\alpha, \boldsymbol{v}^\alpha)$ account for the effect of the αth boss. Specifically, they vanish far from the αth boss and $\boldsymbol{v}^\alpha$ vanishes on the entire smooth surface S_s surrounding the αth boss. On the surface B^α of the αth boss, on the other hand, $\boldsymbol{v}^\alpha$ is such that (2.1) is satisfied. To express this requirement it is convenient to define

$$\boldsymbol{w}^\alpha = \boldsymbol{u}_0 + \sum_{\beta \neq \alpha} \boldsymbol{v}^\beta, \qquad r^\alpha = p_0 + \sum_{\beta \neq \alpha} q^\beta, \tag{2.4}$$

so that, for every $\alpha = 1, 2, \ldots, N$,

$$\boldsymbol{u} = \boldsymbol{v}^\alpha + \boldsymbol{w}^\alpha, \qquad p = q^\alpha + r^\alpha. \tag{2.5}$$

On B^α then $\boldsymbol{v}^\alpha$ satisfies

$$\boldsymbol{v}^\alpha = -\boldsymbol{w}^\alpha. \tag{2.6}$$

In the literature on multiple scattering the fields $\boldsymbol{v}^\alpha$ and $\boldsymbol{w}^\alpha$ are often referred to as the 'scattered' and 'incident' fields respectively.

It should be explicitly noted that each $\boldsymbol{v}^\alpha$ is well defined also inside the other bosses and vanishes on their bases. So as to give a meaning to the decompositions (2.3) everywhere in the domain bounded by the smooth surface S_s, we define $\boldsymbol{v}^\alpha$, q^α to be zero inside the αth boss.

The exact fluid dynamic fields $\boldsymbol{u}$, p are necessarily finite even in the 'thermodynamic'

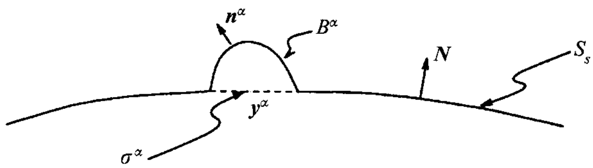

FIGURE 2. Definition of quantities pertaining to an individual boss.

limit in which the number of bosses becomes very large. This consideration is not sufficient to ensure convergence of the summations in (2.3) as the specification of the problem satisfied by $\boldsymbol{v}^\alpha$, q^α involves differential operators that may not commute with the infinite sums. In spite of this, we do not encounter convergence difficulties for two reasons. Firstly, we take the limit on the *differential equations* satisfied by the mean fields, rather than on the *solutions* of these equations. Secondly, we explicitly calculate only the lowest-order correction in the concentration. Were we to attempt a solution at the next order, we would very likely encounter divergencies of the type well known in suspension mechanics (see e.g. Hinch 1977). The origin of this difficulty – the attempt to use one- or two-particle solutions to reconstruct the flow fields in the entire suspension – and a technique to overcome it – renormalization – would probably be applicable to the present problem as well. Further comments on this point can be found in a paper by Rubinstein & Keller (1989) where a decomposition similar to (2.3) is also used.

3. Averaging

Even though the previous problem can be solved exactly by numerical means in some cases, for many applications it is neither useful nor desirable to deal with such a vast amount of information. In these situations, suitable average quantities are of greater practical interest and it is the calculation of such quantities that is our concern here.

We make use of the method of ensemble averaging and consider a large number of rough surfaces, all obtained from S_s by different arrangements of the N bosses. Each arrangement is termed a *configuration* and denoted by $\mathscr{C}^N \equiv (\boldsymbol{Y}^1, \boldsymbol{Y}^2, \ldots \boldsymbol{Y}^N)$, where $\boldsymbol{Y}^\alpha$ is the position of a reference point of the base σ^α of the αth boss (e.g. the centre of symmetry) referred to an arbitrary system of curvilinear coordinates on S_s (figure 2). A particular configuration occurs in the ensemble with a probability $P(\mathscr{C}^N) \equiv P(N)$. Since the bosses are indistinguishable, it is convenient to use the normalization (see e.g. Batchelor 1972)

$$N! = \int \mathrm{d}\mathscr{C}^N P(N) \equiv \int \mathrm{d}^2 Y^1 \int \mathrm{d}^2 Y^2 \ldots \int \mathrm{d}^2 Y^N P(N), \tag{3.1}$$

where an abbreviated notation has been introduced to indicate integration over all possible positions of the bosses over S_s. The restriction to identical bosses can be removed by enlarging the probability space over which P is defined to include additional parameters characterizing the bosses such as size and orientation.

The reduced probability distribution in which the position of K bosses is prescribed is obtained from $P(N)$ by integration:

$$P(K) = \frac{1}{(N-K)!} \int \mathrm{d}\mathscr{C}^{N-K} P(N), \tag{3.2}$$

and satisfies the normalization condition

$$\int \mathrm{d}\mathscr{C}^K\, P(K) = \frac{N!}{(N-K)!}. \tag{3.3}$$

The conditional probability $P(N-K|K)$ for the arrangement $\mathscr{C}^{N-K}$ of $N-K$ bosses, given that K bosses have the arrangement $\mathscr{C}^K$, is defined by

$$P(N-K|K)\,P(K) = P(N). \tag{3.4}$$

From (3.1) and (3.3) one finds the normalization condition

$$\int \mathrm{d}\mathscr{C}^{N-K}\, P(N-K|K) = (N-K)!. \tag{3.5}$$

We can now define the unconditional average of the field $\boldsymbol{u}$ by

$$\langle \boldsymbol{u}\rangle(\boldsymbol{x}) = \frac{1}{N!}\int \mathrm{d}\mathscr{C}^N\, P(N)\,\boldsymbol{u}(\boldsymbol{x}|N), \tag{3.6}$$

where the notation $\boldsymbol{u}(\boldsymbol{x}|N)$ stresses the dependence of the exact field not only on the point $\boldsymbol{x}$, but also on the configuration of the N bosses. In view of the Stokes flow assumption, near the boundary, time dependence can only be parametric and its explicit indication is unnecessary. We introduce conditional averages in which the position of K bosses is held fixed by writing

$$\langle \boldsymbol{u}\rangle_K(\boldsymbol{x}|K) = \frac{1}{(N-K)!}\int \mathrm{d}\mathscr{C}^{N-K}\, P(N-K|K)\,\boldsymbol{u}(\boldsymbol{x}|N), \tag{3.7}$$

with a similar definition for the conditional averages of other quantities of interest.

We now average the problem stated in §2 over the ensemble described by $P(N)$. Upon substitution of the decomposition (2.3) of $\boldsymbol{u}$ into the definition (3.6) one finds

$$\langle \boldsymbol{u}\rangle(\boldsymbol{x}) = \boldsymbol{u}_0 + \sum_{\alpha=1}^{N}\frac{1}{N!}\int \mathrm{d}\mathscr{C}^N P(N)\,\boldsymbol{v}^\alpha. \tag{3.8}$$

In computing the integral it should be kept in mind that $\boldsymbol{v}^\alpha$ has been defined to be zero when $\boldsymbol{x}$ is inside the boss α. Since the bosses are indistinguishable, all the N terms in the sum give the same contribution equal to that of, say, boss α, with α arbitrary, so that

$$\begin{aligned}\langle \boldsymbol{u}\rangle(\boldsymbol{x}) &= \boldsymbol{u}_0 + \frac{1}{(N-1)!}\int \mathrm{d}^2Y^\alpha P(\alpha)\int \mathrm{d}\mathscr{C}^{N-1}\, P(N-1|\alpha)\,\boldsymbol{v}^\alpha \\ &= \boldsymbol{u}_0 + \int \mathrm{d}^2Y^\alpha\, P(\alpha)\langle \boldsymbol{v}^\alpha\rangle_1(\boldsymbol{x}|\alpha),\end{aligned} \tag{3.9}$$

where the definition (3.7) has been used:

$$\langle \boldsymbol{v}^\alpha\rangle_1(\boldsymbol{x}|\alpha) = \frac{1}{(N-1)!}\int \mathrm{d}\mathscr{C}^{N-1}\, P(N-1|\alpha)\,\boldsymbol{v}^\alpha(\boldsymbol{x}|N). \tag{3.10}$$

The problem satisfied by $\langle \boldsymbol{v}^\alpha\rangle_1$ is readily obtained from the exact formulation given in §2 by averaging according to (3.10). In particular, it is readily shown that $\langle \boldsymbol{v}^\alpha\rangle_1, \langle q^\alpha\rangle_1$ satisfy the Stokes equations (2.2) everywhere except inside the αth boss. Furthermore, the ratio of $\langle \boldsymbol{v}^\alpha\rangle_1$ to the incident velocity vanishes at infinity and $\langle \boldsymbol{v}^\alpha\rangle_1$ vanishes on S_s away from the αth boss. On B^α it satisfies

$$\langle \boldsymbol{v}^\alpha\rangle_1 = -\langle \boldsymbol{w}^\alpha\rangle_1. \tag{3.11}$$

In attempting to render the boundary condition (3.11) explicit one encounters the difficulty inherent in all averaging approaches, namely that the mathematical problem for the averaged quantities is not closed. Indeed, upon calculating $\langle \boldsymbol{w}^\alpha \rangle_1$ according to the definition (3.7) with $K = 1$, it is readily found that

$$\langle \boldsymbol{w}^\alpha \rangle_1(\boldsymbol{x}|\alpha) = \boldsymbol{u}_0 + \int \mathrm{d}^2 Y^\beta \, P(\beta|\alpha) \, \langle \boldsymbol{v}^\beta \rangle_2(\boldsymbol{x}|\alpha\beta), \tag{3.12}$$

where now configurations such that $\boldsymbol{x}$ is inside another boss β contribute nothing by definition of $\boldsymbol{v}^\beta$ and $\langle \boldsymbol{v}^\beta \rangle_2$ is given by (3.7) with $K = 2$.

The closure issue will be addressed in the case of a sparse distribution of bosses in §4. We first derive a formal expression for the effective boundary condition valid for arbitrary boss density.

4. The effective boundary condition

We now show that it is possible to derive a formal expression for the effective boundary condition on S_s without solving explicitly the problem posed in the previous section. For this purpose, we use a suitable Green's function representation of the velocity contribution due to the αth boss. Specifically, let $\boldsymbol{G}^S(\boldsymbol{x}, \boldsymbol{y})$ be the Green's function for the Stokes problem vanishing at infinity and also on the smooth surface S_s, and let $\boldsymbol{T}^S$ be the corresponding stress. Then, since $\boldsymbol{v}^\alpha$ vanishes on S_s away from the αth boss, Green's identity is simply (see e.g. Pozrikidis 1992, p. 27),

$$\langle v_j^\alpha \rangle_1(\boldsymbol{x}|\alpha) = \int_{B^\alpha} \mathrm{d}B_y^\alpha \, \langle g_i^\alpha \rangle_1(\boldsymbol{y}|\alpha) \, G_{ij}^S(\boldsymbol{y}, \boldsymbol{x}) - \int_{B^\alpha} \mathrm{d}B_y^\alpha \, \langle v_i^\alpha \rangle_1(\boldsymbol{y}|\alpha) \, T_{ijk}^S(\boldsymbol{y}, \boldsymbol{x}) \, n_k, \tag{4.1}$$

where the integration is over the boss surface B^α. Here in the first integral (single layer) $\boldsymbol{g}^\alpha$ is the traction force at the boundary associated with the velocity field $\boldsymbol{v}^\alpha$ and given by

$$\boldsymbol{g}^\alpha = -q^\alpha \boldsymbol{n} + \boldsymbol{\tau}^\alpha \cdot \boldsymbol{n} \tag{4.2}$$

where $\boldsymbol{\tau}^\alpha$ is the Newtonian viscous stress tensor and $\boldsymbol{n}$ the unit normal into the fluid (figure 2).

Similar to our approach for Laplace's equation (Sarkar & Prosperetti 1995), we note that the 'incident' field $\langle \boldsymbol{w}^\alpha \rangle_1$ is a regular solution of the Stokes equations in the closed domain bounded by the surface B^α of the boss and the underlying portion σ^α of S_s (see figure 2 for a definition of these symbols). Green's identity written at the same point $\boldsymbol{x}$ appearing in (4.1) for the closed surface $\sigma^\alpha \bigcup B^\alpha$ therefore reduces to

$$0 = \int_{B^\alpha} \mathrm{d}B_y^\alpha \, \langle h_i^\alpha \rangle_1(\boldsymbol{y}|\alpha) G_{ij}^S(\boldsymbol{y}, \boldsymbol{x}) - \int_{B^\alpha} \mathrm{d}B_y^\alpha \, \langle w_i^\alpha \rangle_1(\boldsymbol{y}|\alpha) T_{ijk}^S(\boldsymbol{y}, \boldsymbol{x}) n_k, \tag{4.3}$$

where $\boldsymbol{h}^\alpha$ is the traction force at the boundary due to the velocity field $\boldsymbol{w}^\alpha$. The left-hand side vanishes because $\boldsymbol{x}$ is outside the surface $\sigma^\alpha \bigcup B^\alpha$. The first integral does not contain a contribution from σ^α because $\boldsymbol{G}^S$ vanishes by construction for $\boldsymbol{y}$ on S_s, and similarly for the second integral since $\boldsymbol{w}^\alpha = 0$ on S_s from the definition (2.4) and the conditions satisfied by $\boldsymbol{u}_0$ and the $\boldsymbol{u}^\beta$. Equation (4.3) is of course just a special case of Lorentz's reciprocal theorem (Lorentz 1896).

By adding (4.3) to (4.1), the two integrals containing $\boldsymbol{T}^S$ cancel due to the condition

(3.11), and we find

$$\langle v_j^\alpha \rangle_1(\boldsymbol{x}|\alpha) = \int_{B^\alpha} \mathrm{d}B_y^\alpha \, \left[\langle g_i^\alpha \rangle_1(\boldsymbol{y}|\alpha) + \langle h_i^\alpha \rangle_1(\boldsymbol{y}|\alpha) \right] \, G_{ij}^S(\boldsymbol{y}, \boldsymbol{x}) . \tag{4.4}$$

In order to proceed we express $\boldsymbol{G}^S$ near $\boldsymbol{y}^\alpha$ as

$$G_{ij}^S(\boldsymbol{y}, \boldsymbol{x}) = G_{ij}^P(\boldsymbol{y}, \boldsymbol{x}) + F_{ij}(\boldsymbol{y}, \boldsymbol{x}) . \tag{4.5}$$

The first term is the Green's function for a no-slip plane tangent to S_s at $\boldsymbol{y}^\alpha$. The term F_{ij} accounts for the deviation of S_s from this tangent plane. Since the radius of curvature R_s of S_s has been assumed to be large compared with the scale of the bosses, one expects that $|F_{ij}| \ll |G_{ij}^P|$ up to distances from the boss smaller than R_s, but still much greater than a, and in particular in the matching region. The expression for the Green's function $\boldsymbol{G}^P$ for a plane wall was essentially first obtained by Lorentz (1896). An equivalent expression was given by Blake (1971; see also Blake & Chwang 1974). Later Hasimoto & Sano (1980) provided an elegant form which has the advantage of being formally valid also in the two-dimensional case (see §7). This form is

$$G_{ij}^P(\boldsymbol{y}, \boldsymbol{x}) \; = \; G_{ij}(\boldsymbol{y}, \boldsymbol{x}) - G_{ij}(\boldsymbol{y}, \boldsymbol{x}') \pm 2 y_1 \partial_{y_j} G_{i1}(\boldsymbol{y}, \boldsymbol{x}') \mp \frac{y_1^2}{\mu} \partial_{y_j} \Pi_i(\boldsymbol{y}, \boldsymbol{x}') , \tag{4.6}$$

where $\boldsymbol{G}$ is the free-space Green's function (Stokeslet) given by

$$G_{ij}(\boldsymbol{y}, \boldsymbol{x}) = -\frac{1}{8\pi\mu} \left[\frac{\delta_{ij}}{|\boldsymbol{x} - \boldsymbol{y}|} + \frac{(x_i - y_i)(x_j - y_j)}{|\boldsymbol{x} - \boldsymbol{y}|^3} \right] , \tag{4.7}$$

and $\boldsymbol{\Pi}(\boldsymbol{y}, \boldsymbol{x})$ is the corresponding pressure field (formally equal to a potential dipole), namely

$$\boldsymbol{\Pi}(\boldsymbol{y}, \boldsymbol{x}) = -\frac{\boldsymbol{x} - \boldsymbol{y}}{4\pi |\boldsymbol{x} - \boldsymbol{y}|^3} . \tag{4.8}$$

In these relations the index 1 refers to the normal direction into the fluid, $\boldsymbol{x}'$ is the image of $\boldsymbol{x}$ in the plane, the plus sign is for the normal direction $j = 1$ and the minus sign for $j = 2, 3$. (By $\boldsymbol{y}^\alpha$ we indicate the position vector of the centre of σ^α in three-dimensional space; the notation $\boldsymbol{Y}^\alpha$ used earlier refers to the position of the same point expressed in terms of curvilinear coordinates on the surface S_s.)

Equation (4.4) can be simplified considerably if we take the point $\boldsymbol{x}$ in an intermediate range far from the boss on the boss scale, while still close to S_s with respect to the surface radius of curvature, i.e. $a \ll |\boldsymbol{x} - \boldsymbol{y}^\alpha| \ll R_s$. In the language of singular perturbations, this would be the 'matching region', and it is at this point that we explicitly use the postulated separation of scales between the bosses' size and the 'macroscopic' dimensions of S_s. As noted before, in this region the Green's function correction F is small and can be neglected. Hence, upon using the symmetry property $G_{ij}(\boldsymbol{y}, \boldsymbol{x}) = G_{ji}(\boldsymbol{x}, \boldsymbol{y})$, an expansion of $\boldsymbol{G}^P$ in Taylor series in $\boldsymbol{y}$ centred at $\boldsymbol{y}^\alpha$ gives, for $i = 2$, 3 (Blake & Chwang 1974; Pozrikidis 1992, p. 86),

$$\begin{aligned} G_{ji}^P(\boldsymbol{x}, \boldsymbol{y}) &\simeq \frac{3}{2\pi\mu} \frac{(x_i - y_i^\alpha)(x_j - y_j^\alpha)(x_1 - y_1^\alpha)}{|\boldsymbol{x} - \boldsymbol{y}^\alpha|^5} (y_1 - y_1^\alpha) \left[1 + O\left(\frac{a}{|\boldsymbol{x} - \boldsymbol{y}^\alpha|} \right) \right] \\ &\simeq \frac{2}{\mu} T_{ij1}(\boldsymbol{x}, \boldsymbol{y}^\alpha)(y_1 - y_1^\alpha) , \end{aligned} \tag{4.9}$$

where

$$T_{ijk}(\boldsymbol{x}, \boldsymbol{y}) = \frac{3}{4\pi} \frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{|\boldsymbol{x} - \boldsymbol{y}|^5} . \tag{4.10}$$

The corresponding result for $i = 1$ is

$$G^P_{j1}(\boldsymbol{x},\boldsymbol{y}) \simeq -\frac{3}{4\pi\mu}\frac{(x_1-y_1^\alpha)}{|\boldsymbol{x}-\boldsymbol{y}^\alpha|^5}\left[2(x_i-y_i^\alpha)+\delta_{i1}(x_1-y_1^\alpha)-5\frac{(x_1-y_1^\alpha)^2(x_i-y_i^\alpha)}{|\boldsymbol{x}-\boldsymbol{y}^\alpha|^2}\right](y_1-y_1^\alpha)^2. \tag{4.11}$$

This form shows that G^P_{j1} is of the same order as the terms that have been dropped in the expression for G^P_{ji} with $i = 2, 3$ and can therefore be disregarded. It can actually be shown that this term would contribute zero to the final result (4.18).

Upon substituting (4.9) into the integral representation (4.4) we then find

$$\langle v_j^\alpha\rangle_1 = 2\sum_{i=2}^{3}(\Phi_i^\alpha+\Psi_i^\alpha)T_{ij1}(\boldsymbol{x},\boldsymbol{y}^\alpha), \tag{4.12}$$

where

$$\boldsymbol{\Phi}^\alpha(\langle\boldsymbol{v}^\alpha\rangle_1) = \frac{1}{\mu}\int_{B^\alpha}\mathrm{d}B_y^\alpha\,\boldsymbol{N}\cdot(\boldsymbol{y}-\boldsymbol{y}^\alpha)\,\langle\boldsymbol{g}_\parallel^\alpha\rangle_1(\boldsymbol{y}|\alpha), \tag{4.13}$$

$$\boldsymbol{\Psi}^\alpha(\langle\boldsymbol{w}^\alpha\rangle_1) = \frac{1}{\mu}\int_{B^\alpha}\mathrm{d}B_y^\alpha\,\boldsymbol{N}\cdot(\boldsymbol{y}-\boldsymbol{y}^\alpha)\,\langle\boldsymbol{h}_\parallel^\alpha\rangle_1(\boldsymbol{y}|\alpha). \tag{4.14}$$

Here the subscript $\|$ denotes the component parallel to the tangent plane to the smooth surface S_s at $\boldsymbol{y}^\alpha$ and $\boldsymbol{N}$ is the unit normal to S_s oriented into the fluid. Since in (4.12) the summation is only over the components parallel to S_s, $\boldsymbol{\Phi}^\alpha$ and $\boldsymbol{\Psi}^\alpha$ have been defined so that $\Phi_1^\alpha = 0$, $\Psi_1^\alpha = 0$. Furthermore note that, since $\langle\boldsymbol{g}^\alpha\rangle_1$ and $\langle\boldsymbol{h}^\alpha\rangle_1$ are the tractions corresponding to $\langle\boldsymbol{v}^\alpha\rangle_1$ and $\langle\boldsymbol{w}^\alpha\rangle_1$, and are therefore related to these quantities by linear relations, $\boldsymbol{\Phi}$, $\boldsymbol{\Psi}$ are linear functionals of their arguments. It may also be noted that, from (2.5),

$$\boldsymbol{\Phi}^\alpha+\boldsymbol{\Psi}^\alpha = \frac{1}{\mu}\int_{B^\alpha}\mathrm{d}B_y^\alpha\,\boldsymbol{N}\cdot(\boldsymbol{y}-\boldsymbol{y}^\alpha)\,\langle\boldsymbol{f}_\parallel\rangle_1(\boldsymbol{y}|\alpha), \tag{4.15}$$

where $\boldsymbol{f}^\alpha = \boldsymbol{g}^\alpha+\boldsymbol{h}^\alpha$ is the traction force on the αth boss due to the total flow velocity $\boldsymbol{u} = \boldsymbol{v}^\alpha+\boldsymbol{w}^\alpha$. This relation exhibits the direct relation between the perturbation velocity due to the αth boss and the torque acting on the boss.

The result (4.12) may now be inserted into the expression (3.9) for the average field to find

$$\langle u_j\rangle(\boldsymbol{x}) = u_{0j}+2\sum_{i=2}^{3}\int_{S_s}\mathrm{d}^2Y^\alpha\,P(\alpha)(\Phi_i^\alpha+\Psi_i^\alpha)T_{ij1}. \tag{4.16}$$

This relation shows that, at some distance from the rough surface, the effect of the bosses is represented by a suitable distribution of doublets over the surface S_s. We now take an 'inner limit' of (4.16) by letting the field point $\boldsymbol{x}$ approach S_s to find (see e.g. Stakgold 1979, p. 513)

$$\langle u_j\rangle(\boldsymbol{x})|_{S_s} = -P(\boldsymbol{x})(\Phi_j+\Psi_j)+2\sum_{i=2}^{3}\int_{S_s}\mathrm{d}^2Y^\alpha\,P(\alpha)(\Phi_i^\alpha+\Psi_i^\alpha)T_{ij1}, \tag{4.17}$$

where we have used the fact that $\boldsymbol{u}_0$ vanishes on the plane and dropped the superscript α in the first term as $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ here are to be evaluated for the boss centred at $\boldsymbol{x} \in S_s$. In (4.17) the integral is understood as a principal value and vanishes identically if S_s is plane. If the radius of curvature is large, as we have already assumed in dropping the correction F_{ij} to the plane Green's function in (4.5), its contribution will be small and (4.17) gives the effective boundary condition

$$\langle \boldsymbol{u}_{\|} \rangle(\boldsymbol{x}) = -P(\boldsymbol{x})[\boldsymbol{\Phi} + \boldsymbol{\Psi}], \qquad \boldsymbol{x} \in S_s, \tag{4.18}$$

where the subscript $\|$ on $\boldsymbol{u}$ is a consequence of the fact that, by definition, $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ only have components in the tangent plane.

According to this result, the average velocity field evidently does not vanish on S_s in general. This result may be interpreted by noting that, from (4.15), $(-\Phi_3^\alpha - \Psi_3^\alpha, \Phi_2^\alpha + \Psi_2^\alpha)$ are the components in the plane of the torque acting on the boss. The action of the average of this torque is balanced by a vortex sheet that produces a velocity discontinuity.

We proceed similarly for the normal component of $\langle \boldsymbol{v}^\alpha \rangle_1$ for which only the expression (4.11) is needed. After substitution into (3.9), standard techniques enable one to evaluate the inner limit for $\boldsymbol{x}$ approaching the surface with the result

$$\langle \boldsymbol{u}_{\perp} \rangle_1(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in S_s, \tag{4.19}$$

where $\boldsymbol{u}_{\perp}$ denotes the component of $\boldsymbol{u}$ normal to the smooth surface S_s. The contribution given by a regularized integral similar to that in (4.17) has been neglected under the same hypotheses as before.

5. The first-order problem

We now obtain an explicit expression for the effective boundary condition (4.18) in the sparse limit, i.e. to first order in the area fraction covered by the bosses.

We start by noting that, since $\boldsymbol{w}^\alpha$ accounts for the effect of all the other bosses on the one located at $\boldsymbol{y}^\alpha$, $\langle \boldsymbol{w}^\alpha \rangle_1$ is slowly varying near $\boldsymbol{y}^\alpha$ so that, for $\boldsymbol{y}$ on the boss surface B^α, we may write

$$\langle \boldsymbol{w}^\alpha \rangle_1(\boldsymbol{y}|\alpha) = \langle \boldsymbol{w}^\alpha \rangle_1(\boldsymbol{y}^\alpha|\alpha) + [(\boldsymbol{y} - \boldsymbol{y}^\alpha) \cdot \nabla] \langle \boldsymbol{w}^\alpha \rangle_1(\boldsymbol{y}^\alpha|\alpha) + \dots . \tag{5.1}$$

However, since bosses cannot overlap, $\langle \boldsymbol{w}^\alpha \rangle_1$ vanishes by definition on the base of the αth boss so that $\langle \boldsymbol{w}^\alpha \rangle_1(\boldsymbol{y}^\alpha|\alpha) = 0$. For the same reason, only the gradient in the normal direction, $\boldsymbol{N} \cdot \nabla \langle \boldsymbol{w}^\alpha \rangle_1(\boldsymbol{y}^\alpha|\alpha)$, is non-zero and furthermore, since $\nabla \cdot \boldsymbol{w}^\alpha = 0$, $\partial \langle w_1^\alpha \rangle_1 / \partial y_1 = 0$ at $\boldsymbol{y}^\alpha$. As a consequence, upon retaining only the terms shown in (5.1), on B^α we find the following approximate expression for the traction $\boldsymbol{h}^\alpha$ due to the velocity field $\boldsymbol{w}^\alpha$:

$$\langle \boldsymbol{h}_{\|}^\alpha \rangle_1(\boldsymbol{y}|\alpha) \simeq -\langle r^\alpha \rangle_1(\boldsymbol{y}^\alpha|\alpha)\boldsymbol{n}_{\|} + \mu(\boldsymbol{n} \cdot \boldsymbol{N})(\boldsymbol{N} \cdot \nabla)\langle \boldsymbol{w}_{\|}^\alpha \rangle_1(\boldsymbol{y}^\alpha|\alpha), \tag{5.2}$$

where $\boldsymbol{n}_{\|} = \boldsymbol{N} \times (\boldsymbol{n} \times \boldsymbol{N})$ is the component of $\boldsymbol{n}$ parallel to the tangent plane and the first term is the pressure disturbance caused by the other bosses (see (2.4)). Upon substitution into the definition (4.14) of $\boldsymbol{\Psi}$, since

$$\int_{B^\alpha} \mathrm{d}B_y^\alpha \, [\boldsymbol{N} \cdot (\boldsymbol{y} - \boldsymbol{y}^\alpha)] \, \boldsymbol{n}_{\|} = 0, \tag{5.3}$$

the pressure term contributes nothing and

$$\boldsymbol{\Psi}^\alpha(\langle \boldsymbol{w}_\alpha \rangle_1) \simeq V(\boldsymbol{N} \cdot \nabla)\langle \boldsymbol{w}_{\|}^\alpha \rangle_1(\boldsymbol{y}^\alpha|\alpha), \tag{5.4}$$

where the approximation is solely due to the use of (5.1) and the constant V, given by

$$V = \int_{B^\alpha} \mathrm{d}B_y^\alpha \, [\boldsymbol{N} \cdot (\boldsymbol{y} - \boldsymbol{y}^\alpha)] \, (\boldsymbol{N} \cdot \boldsymbol{n}), \tag{5.5}$$

is readily seen to be the (common) volume of the bosses. If L denotes the characteristic length scale for the variation of $\langle \boldsymbol{w}_{\parallel}^{\alpha}\rangle_1$, the relative error in (5.4) is of the order of $(a/L)^2$.

The previous method cannot be used for the calculation of $\boldsymbol{\Phi}^{\alpha}$ because $\langle \boldsymbol{v}^{\alpha}\rangle_1$ is not slowly varying near B^{α}. Rather, the calculation of this quantity is reduced to a canonical problem as follows. Upon using the approximation (5.1), the condition (3.11) satisfied by $\langle \boldsymbol{v}^{\alpha}\rangle_1$ on B^{α} may be written

$$\langle \boldsymbol{v}^{\alpha}\rangle_1(\boldsymbol{x}|\alpha) \simeq -[(\boldsymbol{x}-\boldsymbol{y}^{\alpha})\cdot \boldsymbol{N}]\,(\boldsymbol{N}\cdot\nabla)\langle \boldsymbol{w}_{\parallel}^{\alpha}\rangle_1(\boldsymbol{y}^{\alpha}|\alpha). \tag{5.6}$$

This relation shows that, on the surface of the boss, to this approximation the velocity field induced by the other bosses is a simple shear flow. Since $\langle \boldsymbol{v}^{\alpha}\rangle_1$ satisfies a linear problem that is homogeneous except for this condition, it is evident that it must be expressible in the form

$$\langle \boldsymbol{v}^{\alpha}\rangle_1(\boldsymbol{x}|\alpha) = \boldsymbol{V}^{(2)}(\boldsymbol{x}-\boldsymbol{y}^{\alpha})\left[(\boldsymbol{N}\cdot\nabla)\langle w_2^{\alpha}\rangle_1(\boldsymbol{y}^{\alpha}|\alpha)\right]+\boldsymbol{V}^{(3)}(\boldsymbol{x}-\boldsymbol{y}^{\alpha})\left[(\boldsymbol{N}\cdot\nabla)\langle w_3^{\alpha}\rangle_1(\boldsymbol{y}^{\alpha}|\alpha)\right], \tag{5.7}$$

where the two vectors $\boldsymbol{V}^{(i)}$, $i=2,3$, are the velocity fields that solve the Stokes equations subject to the boundary condition

$$\boldsymbol{V}^{(i)}(\boldsymbol{x}) = -(\boldsymbol{x}\cdot\boldsymbol{N})\,\hat{\boldsymbol{e}}_i \tag{5.8}$$

on the surface of the boss centred at the origin and vanish at infinity and on the plane away from the boss. Here $\hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3$ is a pair of orthogonal unit vectors in the tangent plane. Clearly, the fields $\boldsymbol{V}^{(i)}$ are only dependent on the shape of the boss.

With the expression (5.7) for $\langle \boldsymbol{v}^{\alpha}\rangle_1$, the quantity $\boldsymbol{\Phi}^{\alpha}$ defined in (4.13) may be written in the form

$$V^{-1}\boldsymbol{\Phi}^{\alpha} = \boldsymbol{k}^{(2)}\left[(\boldsymbol{N}\cdot\nabla)\langle w_2^{\alpha}\rangle_1(\boldsymbol{y}^{\alpha}|\alpha)\right]+\boldsymbol{k}^{(3)}\left[(\boldsymbol{N}\cdot\nabla)\langle w_3^{\alpha}\rangle_1(\boldsymbol{y}^{\alpha}|\alpha)\right], \tag{5.9}$$

where

$$\boldsymbol{k}^{(i)} = \frac{1}{V\mu}\int_B \mathrm{d}B_y\,(\boldsymbol{N}\cdot\boldsymbol{y})\,\boldsymbol{\Sigma}_{\parallel}^{(i)}(\boldsymbol{y}) \tag{5.10}$$

is proportional to the integral over the surface of the boss centred at the origin of the surface traction $\boldsymbol{\Sigma}^{(i)}$ associated with the velocity field $\boldsymbol{V}^{(i)}$. Recall that V is the volume of the bosses. Clearly, $(-k_3^{(i)}, k_2^{(i)})$ are the components in the plane of the dimensionless torque on the boss exerted by the flow $\boldsymbol{V}^{(i)}$.

The final result may be expressed in a more compact form upon introducing the two-dimensional tensor

$$\mathscr{K}_{ij} = k_i^{(j)}, \tag{5.11}$$

as, upon substitution of (5.4), (5.9) into the form (4.18) of the effective boundary condition, we then find

$$\langle \boldsymbol{u}_{\parallel}\rangle(\boldsymbol{x}) = -P(\boldsymbol{x})\,V(\mathscr{I}+\mathscr{K})\cdot\left[(\boldsymbol{N}\cdot\nabla)\langle \boldsymbol{w}_{\parallel}\rangle_1(\boldsymbol{x}|\boldsymbol{x})\right], \tag{5.12}$$

where $\mathscr{I}$ is the identity 2-tensor. Upon recognizing that

$$P(\boldsymbol{x})\,V = C(\boldsymbol{x}) \tag{5.13}$$

is the volume occupied by the bosses per unit surface of S_s, we may also write

$$\langle \boldsymbol{u}_{\parallel}\rangle(\boldsymbol{x}) = -C(\boldsymbol{x})\,(\mathscr{I}+\mathscr{K})\cdot\left[(\boldsymbol{N}\cdot\nabla)\langle \boldsymbol{w}_{\parallel}\rangle_1(\boldsymbol{x}|\boldsymbol{x})\right]. \tag{5.14}$$

This is still an incomplete result as the field $\langle \boldsymbol{w}_{\parallel}\rangle_1$ is not known. Since, however, this field appears here multiplied by a quantity of the first order in the boss concentration, it is consistent to use for it an approximation of zero-order accuracy. The classic approximation introduced by Foldy (1945) consists in setting

$$\langle \boldsymbol{w}_{\parallel}\rangle_1(\boldsymbol{x}|\boldsymbol{x}) \simeq \langle \boldsymbol{u}_{\parallel}\rangle(\boldsymbol{x}), \tag{5.15}$$

so that the effective boundary condition (5.14) finally becomes

$$\langle \boldsymbol{u}_{\parallel}\rangle(\boldsymbol{x}) = -C(\boldsymbol{x})\,(\mathscr{I}+\mathscr{K})\cdot\left[(\boldsymbol{N}\cdot\nabla)\langle \boldsymbol{u}_{\parallel}\rangle(\boldsymbol{x})\right]. \tag{5.16}$$

The original no-slip boundary condition is thus transformed to a mixed condition for the tangential velocity. A discussion of this result will be given in §8. Its validity hinges on the assumptions that the boss size is much smaller than the macroscopic length scales, that the flow in the vicinity of the bosses is adequately described by the Stokes equations, and that the bosses are sparsely distributed on average.

In conclusion, we prove that the matrix $\mathscr{K}$ defined in (5.11) is symmetric. For this purpose note that, by (5.8), we have

$$\mathscr{K}_{ij} = \frac{1}{V\mu}\int_B \mathrm{d}B_y\,(\boldsymbol{N}\cdot\boldsymbol{y})\,\boldsymbol{\Sigma}^{(i)}\cdot\hat{\boldsymbol{e}}_j = -\frac{1}{V\mu}\int_B \mathrm{d}B_y\;\boldsymbol{\Sigma}^{(i)}\cdot\boldsymbol{V}^{(j)}(\boldsymbol{y}). \tag{5.17}$$

Since, by definition, $\boldsymbol{V}^{(j)}$ vanishes everywhere other than on the boss B, the integral can be extended to the entire boundary of the problem. The reciprocal theorem for Stokes flow can then be used to replace the integrand in (5.17) by $\boldsymbol{\Sigma}^{(j)}\cdot\boldsymbol{V}^{(i)}(\boldsymbol{y})$ to prove the statement.

6. Hemispherical bosses

The shape of the bosses affects the effective boundary condition through the tensor $\mathscr{K}$ whose evaluation requires the calculation of the Stokes fields $\boldsymbol{V}^{(2)}$, $\boldsymbol{V}^{(3)}$ satisfying (5.8). In general such a solution has to be determined numerically by one of the several effective techniques available (Weinbaum, Ganatos & Yan 1990; Pozrikidis 1992; Kim & Karrila 1991). For the simple case of hemispherical bosses, however, a semi-analytic solution is possible (Price 1985; the earlier work of Hyman 1972 is incorrect).

In view of the high symmetry of the boss shape, it is easy to convince oneself that the tensor $\mathscr{K}$ is isotropic for this case, $\mathscr{K} = k\delta_{ij}$. The constant k is given in Sarkar (1994):

$$\begin{aligned} k &= \frac{3}{2\pi}\int_B x_1\Sigma_2^{(2)}\mathrm{d}B \\ &= \frac{3}{2}\int_0^{\pi/2}(\sin\theta\,\partial_r Q - \sin\theta\, Q + \partial_r\psi)\cos\theta\sin\theta\,\mathrm{d}\theta, \end{aligned} \tag{6.1}$$

where the functions Q and ψ are defined by O'Neill (1968). Performing the integration numerically, one obtains

$$k \simeq \tfrac{3}{2}\times 0.2104 = 0.3156. \tag{6.2}$$

The effective boundary condition (5.16) for a rough surface with sparse hemispherical bosses is then

$$\langle \boldsymbol{u}_{\parallel}\rangle(\boldsymbol{x}) = -1.3156\;C(\boldsymbol{x})\,(\boldsymbol{N}\cdot\nabla\langle \boldsymbol{u}_{\parallel}\rangle)(\boldsymbol{x}), \tag{6.3}$$

with the usual zero normal velocity condition.

7. The two-dimensional case

The preceding analysis can also be applied to the corresponding two-dimensional problem, i.e. a surface with a random distribution of parallel or nearly parallel 'ridges'.

The results found previously hold also in this case with the only difference that the free-space Green's function and the corresponding stresses are given by

$$G_{ij}(\boldsymbol{y},\boldsymbol{x}) = -\frac{1}{4\pi\mu}\left\{-\delta_{ij}\ln|\boldsymbol{x}-\boldsymbol{y}| + \frac{(x_i-y_i)(x_j-y_j)}{|\boldsymbol{x}-\boldsymbol{y}|^2}\right\},$$
$$T_{ijk}(\boldsymbol{y},\boldsymbol{x}) = \frac{1}{\pi}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|\boldsymbol{x}-\boldsymbol{y}|^4}, \tag{7.1}$$

$$\boldsymbol{\Pi}(\boldsymbol{y},\boldsymbol{x}) = -\frac{\boldsymbol{x}-\boldsymbol{y}}{2\pi|\boldsymbol{x}-\boldsymbol{y}|^2}. \tag{7.2}$$

After substitution of these expressions into (4.6) the analysis proceeds as before and one finds, for the tangential velocity, an effective boundary condition similar to (4.18), namely

$$\langle \boldsymbol{u}_\parallel\rangle(\boldsymbol{x}) = -P(\boldsymbol{x})[\boldsymbol{\Phi}+\boldsymbol{\Psi}], \tag{7.3}$$

where

$$\boldsymbol{\Phi}^\alpha(\langle \boldsymbol{v}^\alpha\rangle_1) = \frac{1}{\mu}\int_{B^\alpha} \mathrm{d}B_y^\alpha\, \boldsymbol{N}\cdot(\boldsymbol{y}-\boldsymbol{y}^\alpha)\,\langle \boldsymbol{g}_\parallel^\alpha\rangle_1(\boldsymbol{y}|\alpha), \tag{7.4}$$

$$\boldsymbol{\Psi}^\alpha(\langle \boldsymbol{w}^\alpha\rangle_1) = \frac{1}{\mu}\int_{B^\alpha} \mathrm{d}B_y^\alpha\, \boldsymbol{N}\cdot(\boldsymbol{y}-\boldsymbol{y}^\alpha)\,\langle \boldsymbol{h}_\parallel^\alpha\rangle_1(\boldsymbol{y}|\alpha). \tag{7.5}$$

In the sparse limit, to first order, we again find

$$\boldsymbol{\Psi}^\alpha = A(\boldsymbol{N}\cdot\nabla)\,\langle \boldsymbol{w}_\parallel^\alpha\rangle_1(\boldsymbol{y}^\alpha|\alpha), \tag{7.6}$$

$$\boldsymbol{\Phi}^\alpha = A\boldsymbol{K}\left[(\boldsymbol{N}\cdot\nabla)\,\langle \boldsymbol{w}_\parallel^\alpha\rangle(\boldsymbol{y}^\alpha|\alpha)\right], \tag{7.7}$$

where A is the cross-sectional area of the ridges, assumed to be all equal, and $\hat{\boldsymbol{e}}_2$ a unit vector orthogonal to the normal $\boldsymbol{N}$. Similarly to (5.10), K is given by

$$\boldsymbol{K} = \frac{1}{A\mu}\int_B \mathrm{d}B_y\,(\boldsymbol{N}\cdot\boldsymbol{y})\,\boldsymbol{\Sigma}_\parallel, \tag{7.8}$$

where $\boldsymbol{\Sigma}$ is the traction corresponding to the solution $\boldsymbol{V}$ of the Stokes equations vanishing at infinity, on the plane away from the ridge, and subject, on the ridge, to the condition

$$\boldsymbol{V} = -(\boldsymbol{x}\cdot\boldsymbol{N})\,\hat{\boldsymbol{e}}_2. \tag{7.9}$$

The effective boundary condition on the tangential velocity is

$$\langle \boldsymbol{u}_\parallel\rangle(\boldsymbol{x}) = -C(\boldsymbol{x})(1+K)\ [\boldsymbol{N}\cdot\nabla]\,\langle \boldsymbol{u}_\parallel\rangle(\boldsymbol{x}), \tag{7.10}$$

while the normal velocity vanishes. In this case C is the boss area per unit length and is given by

$$C(\boldsymbol{x}) = P(\boldsymbol{x})A. \tag{7.11}$$

8. Examples

In order to illustrate the physical meaning of (5.16) it is now interesting to consider some specific examples. For simplicity we only consider bosses such that the tensor

$\mathscr{K} = k\mathscr{I}$ is isotropic. We have seen in §6 that this would be the case for hemispherical bosses, but other axisymmetric shapes would also result in an isotropic $\mathscr{K}$.

(i) *Rough sphere.* Our first example is the drag on a uniformly (C = constant) rough sphere with radius S moving with velocity $\boldsymbol{U}$ in a fluid at rest in the Stokes regime. The fluid velocity has the usual form (Landau & Lifshitz 1959, section 20)

$$\langle \boldsymbol{u} \rangle = -\frac{a}{r}\left[\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{U})+\boldsymbol{U}\right]+\frac{b}{r}\left[3\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{U})-\boldsymbol{U}\right]. \tag{8.1}$$

The constants a, b are found by imposing (5.16) and the condition of vanishing of the normal velocity and are

$$a = \frac{3}{4}S\left[1+(1+k)\frac{C}{S}\right], \qquad b = \frac{1}{4}S^3\left[1+3(1+k)\frac{C}{S}\right] \simeq \frac{1}{4}S^3\left[1+(1+k)\frac{C}{S}\right]^3. \tag{8.2}$$

The corresponding drag is

$$\boldsymbol{F} = -6\pi\mu S\left[1+(1+k)\frac{C}{S}\right]. \tag{8.3}$$

These results differ from those for a smooth sphere of radius S simply by the substitution of S by

$$S\left[1+(1+k)\frac{C}{S}\right], \tag{8.4}$$

so that the rough sphere behaves like a smooth one with a slightly bigger radius. To first order in C/S these results coincide with those given by Basset (1888, art. 495). They also are in agreement with the 'inclusion monotonicity' theorems on drag in Stokes flow (see Kim & Karrila 1991, section 2.2.4; Hill & Power 1956).

(ii) *Poiseuille flow.* As another example, consider Poiseuille flow in a uniformly rough tube. We readily find the following result for the mass flow rate Q caused by a pressure drop Δp along a tube of length l and radius R:

$$Q = \frac{\pi}{8}\frac{\rho\Delta p}{\mu l}R^4\left[1-4(1+k)\frac{C}{R}\right] \simeq \frac{\pi}{8}\frac{\rho\Delta p}{\mu l}R^4\left[1-(1+k)\frac{C}{R}\right]^4. \tag{8.5}$$

This relation shows that the effect of the roughness is equivalent to a decrease of the tube radius from the value R that it would have were the bosses removed, to the smaller value

$$R\left[1-(1+k)\frac{C}{R}\right]. \tag{8.6}$$

A similar result has been found by Davis (1993) who solved the two-dimensional Stokes flow in a channel with a periodic roughness distribution.

(iii) *Rough plane.* In situations such as the previous ones the effect of roughness is simply to displace into the fluid by a distance $(1+k)C$ the 'effective' boundary, on which the usual no-slip condition would apply. A qualitatively different phenomenon however is encountered when $\mathscr{K}$ is not isotropic or C is not uniform, as then a force may develop acting on the boundary in the direction orthogonal to that of the flow. To illustrate this point we consider the effect of a spatially non-uniform distribution of bosses on the flow generated by the slow oscillations of a plate in its own plane.

As in the previous examples, we assume $\mathscr{K}$ to be isotropic. Although the problem is time-dependent on the macroscopic length scale $(\nu/\omega)^{1/2}$, where ω is the angular frequency of the oscillations, we assume the bosses' size to be so small that inertia is negligible. Thus, time enters only parametrically in the local problem and it is in this sense that one can speak of an effective boundary condition to be imposed on the macroscopic flow.

A full solution of this problem subject to the boundary condition (5.16) leads to coupled integral equations and is a matter of some complexity. In order to bring out the physical effect of interest here, it is sufficient to use a perturbation approach, writing

$$\langle \boldsymbol{u} \rangle = \boldsymbol{u}^0 + \langle \boldsymbol{u}' \rangle, \qquad \langle p \rangle = p^0 + \langle p' \rangle, \tag{8.7}$$

where

$$\boldsymbol{u}^0 = U \exp\left[-(\mathrm{i}\omega/\nu)^{1/2} x + \mathrm{i}\omega t\right] \hat{\boldsymbol{e}}_2, \qquad p^0 = 0, \tag{8.8}$$

with x the coordinate normal to the plate, is the exact solution for a smooth plate. Here $\hat{\boldsymbol{e}}_2$ is the unit vector in the y-direction along which the oscillations take place. To first order in the effect of roughness the perturbation $\langle \boldsymbol{u}' \rangle$ solves the Stokes equations subject, on the plane $x = 0$, to the approximate boundary condition

$$\langle \boldsymbol{u}'_{\parallel}(\boldsymbol{x}) \rangle = -\overline{C}(\boldsymbol{x}) \frac{\partial \boldsymbol{u}^0}{\partial x} = U (\mathrm{i}\omega/\nu)^{1/2} \overline{C} \hat{\boldsymbol{e}}_2, \tag{8.9}$$

where we have set $\overline{C} = (1+k)C$. Here and in the following we drop $\exp \mathrm{i}\omega t$. The normal component of $\langle \boldsymbol{u}' \rangle$ vanishes on $x = 0$, (4.19), and $\langle \boldsymbol{u}' \rangle \to 0$ for $x \to \infty$.

In order to solve this problem we seek a representation of the perturbation velocity in the form

$$\langle \boldsymbol{u}' \rangle = A\hat{\boldsymbol{e}}_1 + \nabla \times (B\hat{\boldsymbol{e}}_1) - \nabla\phi \tag{8.10}$$

where $\hat{\boldsymbol{e}}_1$ is the unit vector in the x-direction normal to the plane and A, B, ϕ are suitable scalar potentials. From the condition of incompressibility

$$\nabla^2\phi = \frac{\partial A}{\partial x}, \tag{8.11}$$

while, from the vorticity equation (Cortelezzi & Prosperetti 1981),

$$\nabla^2 A - \frac{\mathrm{i}\omega}{\nu} A = 0, \qquad \nabla^2 B - \frac{\mathrm{i}\omega}{\nu} B = 0. \tag{8.12}$$

The effective boundary condition (8.9) leads to

$$\frac{\partial^2 B}{\partial y^2} + \frac{\partial^2 B}{\partial z^2} = U \left(\frac{\mathrm{i}\omega}{\nu}\right)^{1/2} \frac{\partial \overline{C}}{\partial z}, \tag{8.13}$$

$$\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = -U \left(\frac{\mathrm{i}\omega}{\nu}\right)^{1/2} \frac{\partial \overline{C}}{\partial y}, \tag{8.14}$$

while, from the vanishing of the normal velocity, on $x = 0$ we have

$$A - \frac{\partial \phi}{\partial x} = 0. \tag{8.15}$$

Of particular interest is the force per unit area Σ_z acting on the plate in the direction *orthogonal* to the plate's velocity. Its expression is

$$\Sigma_z = -\mu \left[\frac{\partial}{\partial x}\left(\frac{\partial B}{\partial y} + \frac{\partial \phi}{\partial z}\right)\right]_{x=0}. \tag{8.16}$$

It is clear from the structure of the problem that B can be expressed as

$$B = U\left(\frac{\mathrm{i}\omega}{\nu}\right)^{1/2}\int_{-\infty}^{\infty}\mathrm{d}\eta\int_{-\infty}^{\infty}\mathrm{d}\zeta\; G_B(x, y-\eta, z-\zeta)\,\frac{\partial\overline{C}}{\partial\zeta}(\eta,\zeta), \tag{8.17}$$

where the Green's function G_B vanishes for $x \to \infty$ and solves

$$\nabla^2 G_B - \frac{\mathrm{i}\omega}{\nu}G_B = 0, \tag{8.18}$$

subject to

$$\frac{\partial^2 G_B}{\partial y^2} + \frac{\partial^2 G_B}{\partial z^2} = \delta(y)\,\delta(z), \tag{8.19}$$

on $x = 0$. In a similar way we have

$$\phi = -U\left(\frac{\mathrm{i}\omega}{\nu}\right)^{1/2}\int_{-\infty}^{\infty}\mathrm{d}\eta\int_{-\infty}^{\infty}\mathrm{d}\zeta\; G_\phi(x, y-\eta, z-\zeta)\,\frac{\partial\overline{C}}{\partial\eta}(\eta,\zeta), \tag{8.20}$$

$$A = -U\left(\frac{\mathrm{i}\omega}{\nu}\right)^{1/2}\int_{-\infty}^{\infty}\mathrm{d}\eta\int_{-\infty}^{\infty}\mathrm{d}\zeta\; G_A(x, y-\eta, z-\zeta)\,\frac{\partial\overline{C}}{\partial\eta}(\eta,\zeta), \tag{8.21}$$

where the Green's functions G_A, G_ϕ are the solutions of

$$\nabla^2 G_\phi = \frac{\partial G_A}{\partial x}, \qquad \nabla^2 G_A - \frac{\mathrm{i}\omega}{\nu}G_A = 0, \tag{8.22}$$

vanishing at infinity and subject, on the plane $x = 0$, to the conditions

$$G_A - \frac{\partial G_\phi}{\partial x} = 0, \qquad \frac{\partial^2 G_\phi}{\partial y^2} + \frac{\partial^2 G_\phi}{\partial z^2} = \delta(y)\,\delta(z). \tag{8.23}$$

Using the representations (8.17), (8.20), (8.21), the expression (8.16) for the cross-stream force density becomes

$$\Sigma_z = \mu U\left(\frac{\mathrm{i}\omega}{\nu}\right)^{1/2}\int_{-\infty}^{\infty}\mathrm{d}\eta\int_{-\infty}^{\infty}\mathrm{d}\zeta\;\left[\frac{\partial G_\phi}{\partial x} - \frac{\partial G_B}{\partial x}\right]_{x=0}\frac{\partial^2\overline{C}}{\partial\eta\partial\zeta}(\eta,\zeta). \tag{8.24}$$

The Green's functions are readily found by taking a Hankel transform in the (y, z)-plane, e.g.

$$\tilde{G}_B = \int_0^{\infty}\mathrm{d}r\, r\, J_0(kr)\, G_B(x, r), \tag{8.25}$$

where $r = (y^2 + z^2)^{1/2}$. They are

$$\tilde{G}_B = -\frac{1}{2\pi k^2}\exp(-hx), \tag{8.26}$$

$$\tilde{G}_\phi = \frac{1}{2\pi k\,(h-k)}\left[\exp(-kx) - \frac{h}{k}\exp(-hx)\right], \tag{8.27}$$

$$\tilde{G}_A = \frac{\mathrm{i}\omega}{2\pi\nu k^2(h-k)}\exp(-hx), \tag{8.28}$$

where $h = (k^2 + \mathrm{i}\omega/\nu)^{1/2}$. By using these results we find for the force per unit area acting on the plate in the z-direction:

$$\Sigma_z = \frac{\mu U}{2\pi}\left(\frac{\mathrm{i}\omega}{\nu}\right)^{1/2}\int_{-\infty}^{\infty}\mathrm{d}\eta\int_{-\infty}^{\infty}\mathrm{d}\zeta\;\frac{1}{[(y-\eta)^2 + (z-\zeta)^2]^{1/2}}\frac{\partial^2\overline{C}}{\partial\eta\partial\zeta}(\eta,\zeta). \tag{8.29}$$

This relation shows that a non-uniform distribution of the bosses can give rise to a stress in the direction *orthogonal* to the motion of the plate.

As an example, consider roughnesses arranged (on average) in lines at an angle λ with the y-axis, e.g.

$$\overline{C} = C_0 \cos\left[Q(y \sin\lambda + z\cos\lambda) - \gamma\right] , \tag{8.30}$$

where C_0, Q, γ are given constants. This might be a simple model of the irregularities resulting from machining in the direction λ. Then, using polar coordinates to effect the integration in (8.29), one readily finds

$$\Sigma_z = -\tfrac{1}{2}\mu U Q \sin 2\lambda \left(\frac{\mathrm{i}\omega}{\nu}\right)^{1/2} \cos\left[Q(y\sin\lambda + z\cos\lambda) - \gamma\right] . \tag{8.31}$$

This result shows that the effect vanishes when the roughnesses are parallel ($\lambda = 0$) or perpendicular ($\lambda = \frac{1}{2}\pi$) to the direction of motion, and it is a maximum when they are arranged at 45° with it. A similar problem has been treated by Wang for a 'wavy' rough surface (Wang 1978) and a finned surface (Wang 1994) with analogous results. His method exploited the linearity of the problem to study separately the motion of the plate in the directions parallel and orthogonal to the grooves or fins. Such an approach evidently cannot be used here where each surface irregularity is inherently three-dimensional.

One may expect that a similar rough plate falling vertically in a viscous fluid would acquire a non-zero velocity in its own plane normal to the direction of gravity. Other examples of similar effects are roughness-induced secondary motion in laminar pipe flow and roughness-induced coupling between translation and rotation for a body in a flow. Roughnesses of such a shape as to give rise to a non-isotropic $\mathscr{K}$ tensor can produce these phenomena even when they are distributed uniformly.

9. Conclusions

We have studied a particular model of a rough surface in which the roughness can be approximated by bosses randomly distributed over a smooth surface S_s. Our analysis holds under the assumption that the flow in the neighbourhood of the bosses is highly viscous as may happen, e.g., in lubrication or for small bosses. On the other hand, use of our methods and results, e.g. in turbulent flow over a rough surface, would be inappropriate.

We have found that, in an ensemble-average sense, the effect of the roughness can be approximately represented by a partial slip boundary condition on the component of the velocity tangent to S_s. This boundary condition, which is non-local in general, simplifies to a local one for widely separated bosses on a surface with a radius of curvature much larger than the boss size. In this case it is

$$\langle \boldsymbol{u}_\parallel \rangle(\boldsymbol{x}) = -C(\boldsymbol{x})(\mathscr{I} + \mathscr{K}) \cdot \left[(\boldsymbol{N}\cdot\nabla)\langle \boldsymbol{u}_\parallel\rangle(\boldsymbol{x})\right] , \tag{9.1}$$

where $\mathscr{K}$, a 2×2 symmetric tensor solely dependent on the boss shape, is defined by (5.11), (5.10) and $\mathscr{I}$ is the two-dimensional identity tensor. The boss density is measured by C, the boss volume per unit surface area. The normal velocity vanishes. Similar conditions are found in the two-dimensional case.

As expected, when $\mathscr{K}$ is isotropic and the roughnesses are uniformly distributed, the physical meaning of this result is that the usual no-slip condition holds not on S_s but on an effective surface displaced into the fluid by a suitable amount. In the case of Stokes's law for a rough sphere, considered as an example in §8, this circumstance has the effect of increasing the drag over that of a smooth sphere. In this case, for example, the meaning of our ensemble-average result is that, given a batch of rough spheres of nominally equal size, their drag distribution can be determined from the probability distribution of their roughnesses.

As first remarked by Jansons (1988), a boundary condition such as (9.1) gives rise to a qualitatively different phenomenon when $\mathscr{K}$ is not isotropic, namely the direction of slip need not coincide with the direction of shear. Since our derivation also applies when C is a function of x, we reach the same conclusion when $\mathscr{K}$ is isotropic, but C is not uniform over the surface. This effect cannot be revealed by the analyses of the two-dimensional problem that can be found in the literature, by the 'inclusion monotonicity' theorems on drag in Stokes flow (see Kim & Karrila 1991, section 2.2.4; Hill & Power, 1956), nor by methods that rely on a uniform distribution of bosses such as Jansons's (1988). We have demonstrated the implications of this phenomenon on an example in §8 in which a plate oscillating in its own plane is subject to a force with a component perpendicular to the direction of motion.

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